ADJOINING AN IDENTITY TO A RING

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Adjoining an Identity to a Ring", submitted by IAN E. GORMAN in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

It is well known that an identity may be added to any ring not already having one.

The usual way, for a ring R of characteristic zero is to form the set $R_A = \{(r,m) \mid r \in R , m \in Z\}$. R_A is made into a ring by defining appropriate operations, and R is embedded in R_A by the map $r \to (r,0)$.

If R has characteristic $k \neq 0$, we use Z_k (i.e. integers modulo k) in place of Z, and do everything else as we would when $\operatorname{char} R = 0$.

In either case, R is embedded as an ideal of R_A.

This method of adding the identity can be used on all rings, any R-module of the same characteristic is in a natural way a unital R_A module, and several properties of the two rings are closely related.

However, the embedding does have some disadvantages. Even though R is an integral domain, prime or semiprime, R_A may not be. A subring of the integers does not get embedded in the integers, and if R already has an identity, the new ring is larger with a different identity. An example of all these situations except the last is found in the ring of even integers.

Because of these objections we will consider an additional method of adding identity, which is discussed in a paper published by Dlab.

This embedding produces a ring R_B which has closer relationship to R and which carries R as an essential idea.

The price paid for this improvement is that $\ell_R(R)$, the left annihilator of R in R must be zero in order to form R_B .

The statements about integral domains, prime, semiprime, torsion-free, and tidy rings are all to be found in Dlab's paper.

J.L. Dorroh may have been the first to publish the embedding which results in R_A . I do not know if Dlab obtained the embedding which results in R_B from any other source.

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CHAPTER I

THE EMBEDDINGS AND SOME OF THEIR PROPERTIES

We will define the two embeddings and consider some of their properties.

Let R be a ring characteristic k (k may be zero). We define Z_k to be the integers modulo k (the integers if $k = 0$) .

On the set $R_A = \{(r,m) \mid r \in R , m \in Z_k\}$ we define two operations by

$$(r_1,m_1) + (r_2,m_2) = (r_1 + r_2 , m_1 + m_2)$$

$$(r_1,m_1)(r_2,m_2) = (r_1r_2 + m_1r_2 + m_2r_1 , m_1m_2)$$

where addition and multiplication are carried out in the appropriate ring, and m_1r_2 indicates the sum of r_2 taken m_1 times. This is easily shown to make R_A a ring.

R is embedded in R_A by the map $r \to (r,0)$, and we designate R and its image interchangeably as R .

Proposition 1.

Every ideal (one or two sided) of R is an ideal (one or two sided) of R_A .

Proof:

If $(r,0) \in I \underset{r}{\triangleleft} R$ then $(r,0)(s,m) = (rs + mr,0) \in I$.

A similar proof can be given for left and two-sided ideals.

As a special case, R is a two-sided ideal of R .

Remark: Since (r,n) is equivalent to $(0,n)$ modulo R , we see easily that R_A/R is isomorphic to Z_ℓ , where $\ell = \text{char } R$.

Dlab's embedding is defined in this way:

Let R be a ring with $\ell_R(R) = 0$. Define $R_B = R_A/\ell_{R_A}(R)$. Since $\ell_R(R) = 0$, $\ell_{R_A}(R) \cap R = 0$ so that R has an isomorphic image in R_B . We will call this image $\overline{R}$ or, when no confusion arises, R .

Remark: $\ell_{R_B}(R) = 0$. If $\overline{(r,m)}$ is in $\ell_{R_B}(R)$, then $\overline{(r,m)}R = 0$ implying that $(r,m)R \subset \ell_{R_A}(R)$. But then $(r,m)R$ is contained in $\ell_R(R) = 0$, because R is a two-sided ideal of R_A , and (r,m) annihilates R . From this, (r,m) is in $\ell_{R_A}(R)$, i.e. $\overline{(r,m)} = 0$.

We have removed some "extra" integers from R_A by defining R_B . We will define an element $r \in R$ to be a left integer if $rx + mx = 0$ for all $x \in R$ and a fixed $m \in Z$. In defining R_B , we removed

$$\ell_{R_A}(R) = \{(r,m) \mid rx + mx = 0 \text{ , all } x \in R\} ,$$

and in so doing equated $(r,0)$ with $(0,m)$. Since $(0,m)$ is in the center of R_A, we note that every left integer of R is a right integer, and so behaves like an ordinary integer.

Remark: If $r(R) = 0$, we can establish by a symmetric argument that all right integers of R are left integers; we need only define an embedding, of the type discussed by Dlab, by removing $r_{R_A}(R)$ instead of $\ell_{R_A}(R)$.

That this cannot always be done is shown by the ring

$$R = \{\begin{pmatrix} o & a \\ o & b \end{pmatrix} \mid a,b \in 2Z\}$$

in which $\ell_R(R) = 0$ but

$$r_R(R) = \{\begin{pmatrix} o & c \\ o & o \end{pmatrix} \mid c \in 2Z\}.$$

The left integers, of the form $\begin{pmatrix} o & a \\ o & a \end{pmatrix}$, are also right integers, but, in addition, we have right integers of the form $\begin{pmatrix} o & c \\ o & d \end{pmatrix}$, where $c \neq d$, which are not left integers.

Remark: Since R_B is a homomorphic image of R_A, and R is isomorphic to $\overline{R}$, every ideal (or one-sided ideal) of $\overline{R}$ is an ideal (or one-sided ideal) of R_B.

Definition: A right ideal I of R is (right) essential if for each (right) ideal $J \neq 0$ of R, $I \cap J \neq 0$.

Remark: Since every ideal is also a right ideal, we see that right essential ideals are also essential.

Proposition 2.

(a) R is a right essential ideal of R_B .

(b) Every (right) essential ideal of R is a (right) essential ideal of R_B .

Proof: (a)

Let $I \triangleleft_r R_B$ and $0 \neq \overline{(r,m)} \in I$.

Since $\ell_{R_B}(R) = 0$ there exists $\overline{(s,o)} \in R$ such that

$\overline{(r,m)(s,o)} \neq 0$ and is in R since $R \triangleleft R_B$.

Thus $I \cap R \neq 0$.

(b)

Let $I \triangleleft_r R$ be right essential in R , and let $J \neq 0$, $J \triangleleft_r R_B$. $R \cap J \neq 0$ by (a) .

$\Longrightarrow I \cap (R \cap J) \neq 0$ because I is right essential in R .

So $I \cap J \neq 0$ and I is right essential in R_B .

Remark: This shows that R_B is, in a sense, a small extension of R . It is small in another sense also - if R already has an identity, $R_B = R$.

We consider again the integers of R when $\ell_R(R) = 0$. If R has elements acting on R as integers, the corresponding integers generate a subring of Z , and thus there is an element of R acting as "smallest" positive integer in R . This integer, corresponding to the positive generator of the subring in Z , is called the mode of R . If there are no integers in R other than zero,

mode $R = 0$.

Remark: If mode $R = \ell$, (r,ℓ) is in $\ell_{R_A}(R)$ for some $r \neq 0$ in R , so that $\overline{(r,o)} = \overline{(o,\ell)}$. From this we can see easily that R_B/R is isomorphic to Z_ℓ .

Since R_B/R is a homomorphic image of $R_A/R \approx Z_k$ $(k = \text{char } R)$, we see that ℓ divides k , i.e. mode R divides char R . Thus if char R is prime, mode R = char R implying that $R_A = R_B$, or mode $R = 1$ implying that $R_B = R$.

It will be necessary to consider the nilpotent elements and the nilpotent and nil ideals of R_A and R_B .

Lemma:

Z_k has nonzero nilpotent elements iff k is divisible by a nonzero square.

Proof: $(\Longrightarrow)$

If $0 \neq n < k$ is nilpotent modulo k , then k divides some power of n , so that n contains every prime factor of k .

Since $n < k$, k must contain a power of some prime p and p^2 divides k .

$(\Longleftarrow)$

Let $k = p^2 r$. Then $pr \neq 0 \mod k$ but $(pr)^2 = 0 \mod k$.

Corollary: All nilpotent elements of R_A (respectively R_B) are contained in R iff char R (respectively mode R and char R) is square-free.

Proof: $(\Longrightarrow)$

$R_A/R \approx Z_k$, $k = \text{char } R$.

$R_B/R \approx Z_\ell$, $\ell = \text{mode } R$,

and the image of a nilpotent element is nilpotent.

$(\Longleftarrow)$

If n is nonzero nilpotent mod char R (respectively mode R and char R) then (o,n) [respectively $\overline{(o,n)}$] is nilpotent in R_A (respectively R_B).

Proposition 3.

$I \underset{1,2}{\triangleleft} R_A$ is nilpotent $\Longrightarrow I \subset R_A(o,m) + J$ for some nilpotent one or two-sided ideal J of R and a suitable integer m which is nilpotent modulo char R.

Proof:

If $I \subset R$, let $m = 0$ and $J = I$.

If $I \not\subset R$ then char $R = k$ is divisible by a nonzero square. Let $x \in I \sim R$.

Then in $R_A/R \approx Z_k$, $\bar{x}$ is nilpotent.

Let m = product of the distinct primes in k.

Then $\bar{x} \in \langle m \rangle$, the maximal nilpotent ideal of Z_k, generated by m. Therefore $x = y + r$, $y \in R_A(o,m)$, $r \in R$.

Let J be the nilpotent ideal $R \cap (I + R_A(o,m))$

$I \subset J + R_A(o,m)$ because if $(r,n) \in I$, then

$(o,n) \in R_A(o,m)$ and so $(r,o) \in J = R \cap (I + R_A(o,m))$.

Proposition 4.

$I \underset{1,2}{\lhd} R_A$ a nil ideal $\Longrightarrow I \subset R_A(o,m) + J$ for some nil one or two-sided ideal J of R and a suitable integer m which is nilpotent modulo char R .

<u>Proof</u>:

Repeat the argument for proposition 3, making note of the fact that since $R_A(o,m)$ is a two-sided nil ideal

$$1 \text{ nil} \Longrightarrow I + R_A(o,m) \quad \text{is nil} .$$

<u>Remark</u>: Propositions 3 and 4 are also true for R_B . We note particularly that m must still be nilpotent modulo char R .

For example, we have

Proposition 3'.

$I \underset{1,2}{\lhd} R_B$ nilpotent $\Longrightarrow I \subset R_B\overline{(o,m)} + J$ for some nilpotent one or two-sided ideal J of R and a suitable integer m which is nilpotent modulo char R .

<u>Proof</u>:

If $I \subset R$, $J = I$, $m = 0$ will do . If not, consider $x \in I \sim R$ and choose m as before. We have $x = \overline{(r,n)}$

where $\overline{(o,n)} \neq 0$ and is outside of R . $\overline{(r,n)}$ can be nilpotent only if $\overline{(o,n)}$ is nilpotent and this means that n is nilpotent modulo char R , because the "integers" $\overline{(o,i)}$ of R_B form a ring isomorphic to the integers modulo char R .

From this, m divides n , and $x = y + r$, $y \in R_B\overline{(o,m)}$ $r \in R$. We define $J = R \cap (I + R_B\overline{(o,m)})$, and show that $I \subset J + R_B\overline{(o,m)}$ as before.

<u>Remark</u>: We note that the centers of R_A and R_B are formed from the center of R in the same way as R_A and R_B are formed from R .

Whenever R_A or R_B modules are mentioned, they are assumed to be unital. Whenever we speak of making an R-module M into an R_A or R_B module we are assuming M char $R = 0$.

CHAPTER II

THE RELATIONSHIP OF SOME PROPERTIES IN R, R_A AND R_B

We now compare the properties of R_A and R_B with those of R.

Proposition 5.

R an integral domain $\Longleftrightarrow$ R_B an integral domain.

Proof: $(\Longrightarrow)$

Note that since R has no zero divisors, $\ell_R(R) = 0$ so that R_B exists.

R an integral domain $\Longrightarrow$ char $R = 0$ or p prime and so

$$R_B/R \approx Z \quad \text{or} \quad Z_p .$$

Therefore $xy = 0$, $x \neq 0$, $y \neq 0 \Longrightarrow$ exactly one of x and y is in R. Say $x = \overline{(r,o)}$, $y = \overline{(s,n)}$.

Then $\overline{(r,o)(s,n)} = 0 \Longrightarrow \overline{(s,n)} = 0$ since $\ell_{R_B}(R) = 0$ and $\overline{(r,o)} \neq 0$.

But this contradicts $\overline{(s,n)} \neq 0$, so R_B is an integral domain.

$(\Longleftarrow)$

Obvious.

Remark: An example of a ring R which is an integral domain when R_A is not, is $R = 2Z$. In R_A, $(2,-2)$ annihilates R.

<u>Definition</u>: R is a prime ring if, when A and B are ideals of $R, AB = 0 \Longrightarrow A = 0$ or $B = 0$.

<u>Remark</u>: Since $\ell_R(R)$ is an ideal in R , $\ell_R(R) = 0$ in a prime ring and R_B exists.

Proposition 6.

R a prime ring $\Longleftrightarrow R_B$ a prime ring.

<u>Proof</u>: $(\Longleftarrow)$

The implication holds since every R ideal is an R_B ideal.

$(\Longrightarrow)$

Let $A,B \trianglelefteq R_B$, with $AB = 0$.

Then $(A \cap R)(B \cap R) = 0 \Longrightarrow A \cap R = 0$ or $B \cap R = 0 \Longrightarrow A$ or $B = 0$ because R is an essential ideal of R_B .

Therefore R_B is prime.

<u>Remark</u>: The proposition is not generally true if we replace R_B by R_A . For example, $R = 2Z$ is a prime ring of characteristic zero in which $2Z$ and $R_A(2,-2)$ annihilate each other. In a ring of characteristic pq , the ideals $R_A(o,p)$ and $R_A(o,q)$ annihilate each other.

<u>Definition</u>: A ring R is semiprime if it has no nonzero nilpotent ideals.

<u>Remark</u>: Since $\ell_R(R)$ is nilpotent, it is zero in a semiprime ring, so that R_B exists.

Proposition 7.

R semiprime $\Longleftrightarrow$ R_B semiprime.

<u>Proof</u>: ($\Longleftarrow$)

The implication holds since every R ideal is an R_B ideal.

($\Longrightarrow$)

Let A be a nonzero nilpotent ideal of R_B.
Then $A \cap R$ is nilpotent but not zero (since R is essential in R_B) contradicting that R is semiprime.

Proposition 7'.

Provided that char R is square-free,

R semiprime $\Longleftrightarrow$ R_A semiprime.

<u>Proof</u>: ($\Longleftarrow$)

As before.

($\Longrightarrow$)

Let I be a nonzero nilpotent ideal of R_A.
Then $I \subset R$ because char R is square-free.
Therefore I is a nonzero nilpotent ideal of R, contradicting that R is semiprime.

<u>Remark</u>: If char R is not square-free, R_A is not semiprime.

<u>Definition</u>: A ring R is said to be torsion-free if no nonzero right annihilator is essential in R.

<u>Remark</u>: Since R is the right annihilator of $\ell_R(R)$, R_B exists whenever R is torsion-free.

Proposition 8.

R torsion-free $\Longleftrightarrow$ R_B torsion-free.

<u>Proof</u>: $(\Longrightarrow)$

Let $0 \neq I \underset{r}{\lhd} R_B$ be an essential right annihilator in R_B and let $0 \neq x$ in R_B such that $xI = 0$.

$0 \neq R_B\, x \underset{r}{\lhd} R_B \Longrightarrow R_B\, x \cap R \neq 0$ (R is right essential in R_B) .

$\Longrightarrow$ there exists $y \neq 0$ in R such that $yI = 0$.

But $I \cap R \neq 0$ (R is right essential in R_B) and is essential in R . Since $I \cap R \subset r_R(y)$, $r_R(y)$ is an essential right annihilator in R of $y \neq 0$.

This contradicts that R is torsion-free.

$(\Longleftarrow)$

Let $I \underset{r}{\lhd} R$ be an essential right annihilator of $y \neq 0$ in R .

Then I is essential in R_B by proposition 2 .

Since $I \subset r_{R_B}(y)$, $r_{R_B}(y)$ is essential in R_B , contradicting that R_B is torsion-free.

<u>Definition</u>: An ideal I is said to be uniform, if when A,B are nonzero ideals contained in I , $A\,B \neq 0$.

<u>Definition</u>: A ring is said to be tidy if it contains an essential direct sum of uniform ideals.

Proposition 9.

R tidy $\Longleftrightarrow$ R_B tidy.

<u>Proof</u>: ($\Longrightarrow$)

An essential direct sum of right ideals in R is a right ideal of R and is therefore essential in R_B by proposition 2.

($\Longleftarrow$)

Let $J = \dot{\sum} U_i$ be an essential direct sum in R_B of the nonzero uniform right ideals U_i.

Suppose $\dot{\sum} (U_i \cap R)$ is not a maximal direct sum in R.

Then there exists $I \underset{r}{\triangleleft} R$ such that $I \dot{+} \dot{\sum} (U_i \cap R)$ is a direct sum. But then $J = I + \sum U_i$ must also be a direct sum, which is a contradiction. Otherwise, $x + \sum u_i = 0$, $x \in I$, $u_i \in U$ with, say, some $u_j \neq 0$. Then there exists $r \in R$ such that $u_j r \neq 0$ (since $\ell_{R_B}(R) = 0$) implying that $x r + \sum u_i r = 0$ where $u_j r \neq 0$. Since $x r \in I$ and $u_i r \in U_i \cap R$, this contadicts that $I + \dot{\sum} U_i \cap R$ is a direct sum.

Since $\dot{\sum} U_i \cap R$ is a maximal direct sum of right R ideals it is right essential in R, and since each U_i is uniform, so is each $U_i \cap R$.

Therefore R is tidy.

<u>Definition</u>: A right R module M is irreducible if $MR \neq 0$ and its only submodules are 0 and M.

<u>Remark</u>: Since every irreducible right R module M is isomorphic to R/I where I is a maximal right ideal of R (Herstein, p. 8), irreducible R modules are also R_B modules.

<u>Definition</u>: A right R module M is faithful if for any $r \in R$, $Mr = 0 \implies r = 0$.

<u>Definition</u>: A ring R is right primitive if it has a faithful irreducible right module.

Bergman has shown that a right primitive ring need not be left primitive.

<u>Remark</u>: If a ring R is either right or left primitive, R_B can be formed because primitive rings are prime (Herstein, p. 42) so that $\ell_R(R) = 0$.

Proposition 10.

R right primitive $\iff R_B$ right primitive.

<u>Proof</u>: $(\implies)$

Let M be a faithful irreducible right R module and consider it as an R_B module.

$M\overline{(r,n)} = 0 \implies M\overline{(r,n)(r',o)} = 0$ for each $r' \in R$.

$\implies \overline{(r,n)(r',o)} = 0$ for each $r' \in R$, since $R \triangleleft R_B$

$\implies \overline{(r,n)} = 0$ since $\ell_{R_B}(R) = 0$. So M is a faithful

R_B module.

Let $mR_B \neq M$ for some $m \in M$. Then $mR_B = 0$ since mR_B is an R-submodule of M, which is irreducible over R. Since $MR_BR = MR \neq 0$, we have $MR_B \neq 0$. Therefore M is irreducible over R_B and R_B is primitive.

$(\Longleftarrow)$

Let R_B be primitive, M a faithful irreducible R_B-module. Then $R \subset R_B = M$ is a faithful R-module and $MR \neq 0$. Let $mR \neq M$, then $mR = 0$ since mR is an R_B - submodule of M, which is irreducible over R_B.

Proposition 11.

R left primitive $\Longleftrightarrow$ R_B left primitive.

Proof: $(\Longrightarrow)$

Let M be a faithful irreducible left R-module and consider it as an R_B module. $\overline{(r,n)}M = 0 \Longrightarrow \overline{(r,n)(r',o)}\, M = 0$ for each $r' \in R \Longrightarrow \overline{(r,n)(r',o)} = 0$ $(R \lhd R_B) \Longrightarrow \overline{(r,n)} = 0$ $(\ell_{R_B}(R) = 0)$. The rest of the proof parallels that of the previous proposition.

Remark: In the above proofs, we have seen that M an irreducible R-module $\Longrightarrow$ M an irreducible R_B module.

Definition: A right ideal of R is said to be irreducible if it is irreducible as a right R-module.

<u>Definition</u>: The socle of R (soc R) is the sum of all the irreducible right ideals of R.

Proposition 12.

soc R = soc R_B if R_B exists.

<u>Proof</u>: ($\supset$)

Let I be an irreducible right ideal of R_B.

$I \cap R \neq 0$ (R is right essential in R_B) $\Longrightarrow I \cap R = I \Longrightarrow I \subset R$. So soc $R_B \subset$ soc R.

(Note that since $\ell_R(R) = 0$, I is actually an irreducible R ideal because $IR \neq 0$).

($\subset$)

I an irreducible right R ideal $\Longrightarrow I$ an irreducible right R_B ideal. (from remark following proposition 11).

<u>Definition</u>: A ring R is semiperfect if it is completely reducible, (i.e. R = soc R), and whenever $x^2 = x \bmod J(R)$ (the Jacobson radical of R), R contains an idempotent which is equivalent to $x \bmod J(R)$.

<u>Remark</u>: Since soc R_B = soc R, R_B can never be semiperfect unless $R_B = R$, that is, unless R already has an identity.

<u>Definition</u>: A ring R is perfect if every strictly descending chain of principal right ideals is finite.

Proposition 13.

R perfect, char $R \neq 0 \Longleftrightarrow R_A$ perfect.

Proof: ($\Longleftarrow$)

R is obviously perfect and if char R = 0 , the chain of ideals $(0,2^n)R_A$, where $n = 1,2,\ldots$, does not terminate.

($\Longrightarrow$)

Let $I_1 \supset I_2 \supset \ldots$ be a strictly descending chain of principal right ideals. Since R_A/R is artinian when char $R \neq 0$, $I_n \supset R$ for some n and so is generated by $(r,o) \in R$ since R_A has identity. Because $(r,o)(s,n) = (r,o)(s,o) + n(r,o)$, this means that I_n (and all of the following ideals in the chain) is principal in R . Since R is perfect and $Ij \subset R$ for $j \geq n$, the chain terminates.

Remark: If R_B exists, the same argument shows that R perfect, mode $R \neq 0 \Longrightarrow R_B$ perfect.

Definition: A ring R is Noetherian if every strictly ascending chain of right ideals is finit.

Definition: A ring R is artinian if every strictly descending chain of right ideals is finite.

If R_A is Noetherian (artinian) then R_A/R , being a homomorphic image, is also Noetherian (artinian), and R is Noetherian (artinian) because every R-ideal is an R_B-ideal.

A similar statement holds for R_B .

Conversely, if R_A/R and R are Noetherian (artinian) then R_A is Noetherian (artinian) and a similar statement is true for R_B . This statement can be proven by showing that two R_A - ideals I and J are equal iff $I \cap R = J \cap R$ and $(I+R)/R = (J+R)/R$, as Lang (p. 143) does when proving a similar statement for Noetherian modules.

From all this we have the following propositions:

Proposition 14.

R Noetherian $\Longleftrightarrow$ R_A (and R_B) Noetherian.

Proof:

R_A/R and R_B/R are always Noetherian.

Proposition 15.

R artinian, char $R \neq 0$ $\Longleftrightarrow$ R_A artinian

R artinian, mode $R \neq 0$ $\Longleftrightarrow$ R_B artinian

Proof:

R_A/R is artinian iff char $R > 0$.

R_B/R is artinian iff mode $R > 0$.

If M is a right R-module, it is also a unital R_A module, provided M char $R = 0$.

<u>Definition:</u> If M and N are R-modules, $\mathrm{Hom}_R(M,N)$ is the set of homomorphisms from M to N.

It is easily seen that $\mathrm{Hom}_R(M,N) = \mathrm{Hom}_{R_A}(M,N)$, when $M \operatorname{char} R = 0$ and $N \operatorname{char} R = 0$.

<u>Definitions:</u> A right R-module M is projective if for modules N and K, when $f : N \to K$ is an epimorphism and $g : M \to K$ is a homomorphism, there exists a homomorphism $h : M \to N$ such that $g = f o h$, that is $g(m) = f[h(m)]$ for $m \in M$.

<u>Definition:</u> A right R-module M is injective if for modules N and K, when $f : N \to K$ is a monomorphism and $g : N \to M$ is a homomorphism, then there exists a homomorphism $h : K \to M$ such that $g = h o f$.

Since $\mathrm{Hom}_R(M,N) = \mathrm{Hom}_{R_A}(M,N)$, we see that M is a projective (injective) R-module iff M is a projective (injective) R_A-module.

If M is a right R-module and $M\ell_{R_A}(R) = 0$ (i.e. $Mo = o$ where the first zero is that of R_B), then M is a unital R_B module, and, as before, M is a projective (injective) R-module iff M is a projective (injective) R_B module.

<u>Definition:</u> $N(R)$ = Noether radical of R = the ideal sum of all nilpotent two-sided ideals of R.

Proposition 16.

$N(R_A) = N(R) + R_A(o,m)$, where $m = 0$ if char R is square free, and m is the ~~nilpotent~~ product of all the distinct primes of char R otherwise.

<u>Proof</u>: $(\subset)$

$I \triangleleft R_A$ a nilpotent ideal $\Longrightarrow$ (by prop. 3) $I \subset J + R(o,m)$ where $J \triangleleft R$ is nilpotent and m is chosen as in the above statement.

Therefore $I \subset N(R) + R_A(o,m)$.

$(\supset)$

$N(R) \subset N(R_A)$, and if m is chosen as above $R_A(o,m)$ will be a nilpotent ideal.

<u>Definition</u>: $U(R)$ = upper nil radical of R .

= ideal sum of all nil two-sided ideals in R .

Proposition 17.

$U(R_A) = U(R) + R_A(o,m)$ where m is chosen as in proposition 16.

<u>Proof</u>: $(\subset)$

$I \triangleleft R_A$ a nil ideal $\Longrightarrow$ $I \subset J + R(o,m)$ by proposition 4, where $J \triangleleft R$ is nil and m is chosen as above.

Therefore $I \subset U(R) + R_A(o,m)$.

$(\supset)$

$U(R) \subset U(R_A)$, and if m is chosen as above $R_A(o,m)$ will be a nil ideal.

<u>Definition</u>: An ideal is locally nilpotent if every finitely generated subring is nilpotent.

Proposition 18.

If $I \triangleleft R_A$ is locally nilpotent then $I \subset J + R_A(o,m)$ where $J \triangleleft R$ is locally nilpotent and m is nilpotent modulo char R .

<u>Proof</u>:

We see this by noting that the image of I in $R_A/R \approx Z_k$ must be nilpotent, so that $I \subset R + R_A(o,m)$ for some m nilpotent modulo char R (choose m as in proposition 3) , and defining $J = \{r \in R \mid (r,n) \in I$ for some $n\}$.
We have $J \triangleleft R$ and it is locally nilpotent because $\langle\{r_i \mid (r_i,n_i) \in I , i = 1 \dots N\}\rangle$ is contained in $\langle\{(r_i,n_i) , i = 1 \dots N\}\rangle + \langle\{(o,m_i) \mid i = 1 \dots N\}\rangle$ which is the sum of two nilpotent ideals.
We can make a similar statement in R_B , as we would for propositions 3 and 4.

<u>Definition</u>: $L(R)$ = the Levitzki radical of R
= the ideal sum of the two-sided locally nilpotent ideals of R .

Proposition 19.

$L(R_A) = L(R) + R_A(o,m)$ where m is chosen as in proposition 16.

Proof: $(\subset)$

by definition of $L(R_A)$

$(\supset)$

$I \triangleleft R_A$ locally nilpotent $\Longrightarrow$ $I \subset J + R_A(o,m)$ where $J \triangleleft R$ is locally nilpotent, and m is chosen as above, so we have $I \subset L(R) + R_A(o,m)$.

Definition: In a ring R , an element x is right quasi-regular if there exists $y \in R$ such that $x + y + xy = 0$.

Remark: Any nilpotent element is right quasi-regular; if $x^m = 0$, we let $y = -x + x^2 - x^3 + \ldots + (-1)^{m-1}x^{m-1}$ to get $x + y + xy = 0$.

Definition: $J(R)$ = the Jacobson radical of R

= the maximal right quasi-regular ideal of R .

Remark: Herstein (p. 9) defines $J(R)$ to be the intersection of the annihilators of all irreducible right R-modules, or, if R has no such modules, to be R . He then shows the two definitions to be equivalent.

Definition: If $J(R) = 0$, R is said to be semisimple.

Proposition 20.

$J(R_A) = J(R) + R_A(o,m)$ where m is chosen as in proposition 16.

<u>Proof</u>:

We note from Herstein (p. 16) that $R \lhd R_A \Longrightarrow J(R) = J(R_A) \cap R$.

Case 1. char $R = 0$

$R_A/R \approx Z$ is semisimple $\Longrightarrow J(R_A) \subset R$

$\Longrightarrow J(R_A) = J(R)$ because $J(R) = J(R_A) \cap R$.

Case 2. char $R \neq 0$

($\supset$)

$R_A(o,m)$ nilpotent $\Longrightarrow$ right quasi-regular

$\Longrightarrow R_A(o,m) \subset J(R_A)$

($\subset$)

In $R_A/R \approx Z_k$, we have $N(Z_k) = J(Z_k)$ because Z_k is right artinian (Ribenboim, p. 16).

Therefore $J(R_A) \subset R + R_A(o,m)$. So we have

$$\begin{aligned} J(R_A) &= J(R_A) \cap (R + R_A(o,m)) \\ &= (J(R_A) \cap R) + R_A(o,m) \quad \text{since } R_A(o,m) \subset J(R_A) \\ &= J(R) + R_A(o,m) . \end{aligned}$$

<u>Definition:</u> $I \lhd R$ is a prime ideal if $A \lhd R$, $B \lhd R$, $AB \subset I \Longrightarrow A \subset I$ or $B \subset I$.

<u>Definition</u>: $P(R)$ = prime radical of R = the intersection of all prime ideals of R .

<u>Definition</u>: An element a of R is strongly nilpotent if every sequence $\{a_i\}$ such that $a_{i+1} \in a_i R a_i$ is ultimately zero, where $a_o = a$.

Lambek (p. 56) shows that the prime radical is the set of all strongly nilpotent elements. Although he assumes all rings have identity, this particular proof does not use the assumption.

Proposition 21.

$P(R_A) = P(R) + R_A(o,m)$ where m is chosen as in proposition 16.

<u>Proof</u>: ($\supset$)

$R_A(o,m)$ consists of strongly nilpotent elements if (o,m) is nilpotent since (o,m) is in the center of R, so that $R_A(o,m) \subset P(R_A)$. Since all R ideals are R_A ideals, every prime ideal of R_A intersects R in a prime ideal, so that

$$\cap\{I \triangleleft R \mid I \text{ prime}\} \subset \cap\,\{I \triangleleft R_A \mid I \text{ prime}\}$$

i.e. $P(R) \subset P(R_A)$.

($\subset$)

$P(R_A) \subset R + R_A(o,m)$ because strongly nilpotent elements are nilpotent.

$P(R_A) \cap R \subset P(R)$ because elements of R which are strongly nilpotent in R_A will be strongly nilpotent in R. From this

$$P(R_A) = P(R_A) \cap (R + R_A(o,m))$$

$$= (P(R_A) \cap R) + R_A(o,m) \quad \text{since} \quad R_A(o,m) \subset P(R_A)$$

$$\subset P(R) + R_A(o,m) \ .$$

Remark: Propositions 16, 20 and 21 are easily proven for R_B by substituting the use of proposition 3' for that of proposition 3.

In the same way, proving proposition 4 for R_B enables us to prove proposition 17 for R_B and proving proposition 18 for R_B enables us to prove proposition 19 for R_B .

BIBLIOGRAPHY

1. Bergman, G. "A ring primitive on the right but not on the left", Proc. Amer. Math. Soc., 15, pp. 473-475, 1964.

2. Brown, Bailey and McCoy, Neal. "Rings with unit elements which contain a given ring", Duke Math. J., 13, pp. 9-20, 1946.

3. Dlab, Vlastimil. "Matrix representation of torsion free rings", Czech. Math. J. 19 (94) pp. 284-298, 1969.

4. Dorroh, J.L. "Concerning adjunctions to algebras", Bull. Amer. Math. Soc. 38, pp. 85-88, 1932.

5. Herstein, Israel N. "Noncommutative Rings", Math. Assn. of America, 1968.

6. Lambek, Joachim. "Lectures on Rings and Modules", Blaisdell, 1966.

7. Lang, Serge. "Algebra", Addison-Wesley, 1965.

8. Ribenboim. "Rings and Modules". Interscience, 1969.